Uitwerking Tentamen Wiskunde VI, 18 juni 1992

1. (i) $z=0$: $f(z)=\frac{e^{2iz}-1}{z}\cdot\frac{z}{\sin z}\cdot\frac{1}{z}$. De eerste 2 factoren hebben in $z=0$ een ophefbare singulariteit en gaan naar $2i$ resp. 1, als $z\to 0$. Dus $f(z)$ heeft 1ste orde pool in $z=0$ (met residu $2i$)

$z=\pi$: $f(z)=\frac{e^{2i(z-\pi)}-1}{z-\pi}\cdot\frac{z-\pi}{\sin z}\cdot\frac{1}{z}$. De eerste 2 factoren hebben in $z=\pi$ een ophefbare singulariteit en dus analytisch in $z=\pi$, de derde factor is analytisch in $z=\pi$. Dus $f(z)$ heeft in $z=\pi$ een ophefbare singulariteit

(ii) Er geldt $e^{z^2}=\sum_0^\infty z^{2k}/k! \Rightarrow \frac{1}{2\pi i}\oint_{|z|=1} e^{z^2}/z^{n+1}\,dz = \begin{cases}0, & n=-1,-2,\dots\\ \frac{1}{k!}, & n=2k\\ 0, & n=2k+1\end{cases}\Big\} k=0,1,2,3\dots$

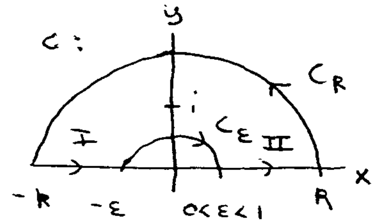
Stel in linkerlid $z=e^{i\varphi}$ dan geldt

li. lid $=\frac{1}{2\pi i}\int_0^{2\pi}\left(e^{e^{2i\varphi}}/e^{(n+1)i\varphi}\right) ie^{i\varphi}d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{EXP}[e^{2i\varphi}-ni\varphi]\,d\varphi$

$=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{EXP}[\cos 2\varphi + i(\sin 2\varphi - n\varphi)]\,d\varphi = \frac{1}{2\pi}\int_0^{2\pi} e^{\cos 2\varphi}\cos(\sin 2\varphi - n\varphi)\,d\varphi$

$+\frac{i}{2\pi}\int_0^{2\pi} e^{\cos 2\varphi}\sin(\sin 2\varphi-n\varphi)\,d\varphi \Rightarrow$ integraal $=\frac{2\pi}{k!}$ als $n=2k$, $k=0,1,2,\dots$ en $=0$ voor alle andere $n\in\mathbb{Z}$

2 C:



(i) Residuenst. $\Rightarrow$ ① $\int_C \frac{\sqrt z}{z^2+1}dz = 2\pi i \operatorname{Res}_{z=i}$

($\sqrt z=\sqrt x$, $z=x>0$)

re lid ① $=2\pi i\lim_{z\to i}\frac{(z-i)e^{\frac12\log z}}{(z-i)(z+i)} = 2\pi i\,\frac{e^{\frac12\log i}}{2i}$

$=\pi e^{\frac14\pi i} = \pi\,\frac12\sqrt2\,(i+1)$, li lid ① $=\int_I+\int_{II}+\int_{C_\varepsilon}+\int_{C_R}$ waarin:

$\int_{II} = (z=x,\ x:\varepsilon\to R,\ dz=dx,\ \sqrt z=\sqrt x) = \int_\varepsilon^R \frac{\sqrt x}{1+x^2}dx$

$\int_I = (z=-x,\ x:R\to\varepsilon,\ dz=-dx,\ \sqrt z = i\sqrt x) = -i\int_R^\varepsilon \frac{\sqrt x}{1+x^2}dx = i\int_\varepsilon^R\frac{\sqrt x}{1+x^2}dx$

$\left|\int_{C_\varepsilon}\right| = (z=\varepsilon e^{i\varphi}\ \ \varphi:\pi\to 0,\ dz=\varepsilon i e^{i\varphi},\ |\sqrt z|=\sqrt\varepsilon,\ |1+z^2|\ge 1-|z|^2=1-\varepsilon^2)$

$=\left|\int_\pi^0 \frac{\sqrt z}{1+z^2}\Big|_{z=\varepsilon e^{i\varphi}} i\varepsilon e^{i\varphi}d\varphi\right| \le \int_0^\pi |\quad|\,d\varphi \le \int_0^\pi\frac{\sqrt\varepsilon}{1-\varepsilon^2}\,d\varphi$

$=\frac{\pi\varepsilon^{3/2}}{1-\varepsilon^2}\to 0$, $\varepsilon\downarrow 0$;

$\left|\int_{C_R}\right| = (z=Re^{i\varphi},\ \varphi:0\to\pi,\ dz=Rie^{i\varphi}d\varphi,\ |\sqrt z|=\sqrt R,\ |1+z^2|\ge|z|^2-1=R^2-1)$

$=\left|\int_0^\pi\frac{\sqrt z}{1+z^2}\Big|_{z=Re^{i\varphi}} iRe^{i\varphi}d\varphi\right|\le\int_0^\pi|\quad|\,d\varphi\le\int_0^\pi\frac{\sqrt R}{R^2-1}R\,d\varphi = \pi\,\frac{R^{3/2}}{R^2-1}$

$\to 0$, $R\to\infty$. Uit ① met $\varepsilon\downarrow0$, $R\to\infty$ volgt $(1+i)\int_0^\infty\frac{\sqrt x}{1+x^2}dx=\frac{\pi}{\sqrt2}(i+1)$

(ii) $0 \overset{\text{Residuenst.}}{=} \int_C\frac{1-e^{iz}}{z^2}dz = \int_I+\int_{II}+\int_{C_\varepsilon}+\int_{C_R} = \int_{-R}^{-\varepsilon}\frac{1-e^{ix}}{x^2}dx+\int_\varepsilon^R\frac{1-e^{ix}}{x^2}dx + (-\pi i)\operatorname{Res}_{z=0}$

$+\int_{C_R} \to PV\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2}dx - \pi = 0$

$\operatorname{Res}_{z=0} = \frac{z(1-e^{iz})}{z^2}\Big|_{z\to0} = \frac{1-e^{iz}}{-iz}\cdot -i\Big|_{z\to0} = -i$ $\qquad \int_{C_R}\to0,\ R\to\infty$

Neem reële delen : $\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}dx=\pi \Rightarrow \int_0^\infty\frac{1-\cos x}{x^2} = \pi/2$